NOTES AND CORRESPONDENCE

On Forced Barotropic Vorticity Oscillations

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ABSTRACT

Forced, nonresonant barotropic response at low frequencies ($\omega \ll f$) and large scales ($L \sim f/\beta$) can be written in terms of a streamfunction, which is similar to the quasigeostrophically derived streamfunction. However, the "nearly equilibrium" forced vorticity equation is valid on the planetary length scale and is influenced not only by the vortex stretching induced by the driving mechanism (tides, atmospheric pressure, or Ekman-pumping displacement) but also by β coupling to the divergent velocity field of the nearly equilibrium response. A similar result follows for topographic coupling, albeit on the topographic length scale.

1. Introduction

Under the assumptions of shallow-water theory (Miles 1974), linear barotropic, forced oceanic motion satisfies

$$u_t - fv = -g\eta_x + Z_x - ru \qquad (1.1a)$$

$$v_t + fu = -g\eta_y + Z_y - rv \qquad (1.1b)$$

$$\eta_t + (uH)_x + (vH)_y = 0, \qquad (1.1c)$$

where we include constant linear damping, proportional to r, and low-frequency forcing (Gill 1982, pp. 336–340)

$$Z \equiv g\bar{\eta} - \frac{p_a}{\rho_0} + \int^t w_{\text{Ekman}} dt' \qquad (1.2)$$

due to long-period tides, atmospheric pressure disturbances, and Ekman-pumping displacement,[1] respectively. The remaining notation is conventional.

In the following section, we demonstrate that, under the assumption that the ocean response is nonresonant (or, equivalently stated, nearly equilibrium, nearly inverted barometer, or nearly isostatic), the rotational response to low-frequency ($\omega/f \ll 1$), large-scale ($L \sim f/\beta$) forcing can be understood in terms of a vorticity equation which is driven not only by the vortex stretching induced by the forcing mechanism, but also by β coupling to the divergent velocity field of the "nearly equilibrium" response.

[1] Thus, W_{Ekman} corresponds to the specified vertical velocity at the base of a thin, frictional surface layer.

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For simplicity, this derivation retains Cartesian geometry, although the analogous derivation for spherical geometry is straightforward. Owing to the original motivation for this research, which was to clarify the nature of oceanic long-period tidal response (Miller 1986, chapter 3), we designate the forcing mechanism as "tidal" in the following two sections.

2. Asymptotic expansion

For the case of a flat bottom, we manipulate (1.1) into the standard continuity, vorticity, and divergence equations, written respectively, as

$$\partial_t \eta = H\nabla^2\phi, \qquad (2.1a)$$

$$(\nabla^2\partial_t + \beta\partial_x + r\nabla^2)\psi = \nabla\cdot(f\nabla\phi), \qquad (2.1b)$$

$$(\nabla^2\partial_t + \beta\partial_x + r\nabla^2)\phi = -\nabla\cdot(f\nabla\psi) + g\nabla^2\eta - g\nabla^2\bar{\eta}, \qquad (2.1c)$$

where ψ represents the velocity streamfunction, ϕ the velocity potential, η the deflection of the free surface from rest, $\bar{\eta}$ the "self-consistent"[2] equilibrium tidal forcing (Agnew and Farrell 1978), and $\beta(y) \equiv df/dy$. The condition of no velocity transport normal to the ocean-basin boundary is satisfied by the convenient (but ad hoc) assumptions that $\nabla\phi\cdot\mathbf{n} = 0$, where $\mathbf{n} \equiv (n_1, n_2)$ is unit normal to the boundary, and $\psi = 0$ along the boundary. Introduce into (2.1) the scalings,

$$(\eta, \bar{\eta}) \sim F(\eta, \bar{\eta}), \quad \psi \sim S\psi, \quad \phi \sim P\phi,$$

[2] Self-consistent forcing implies that if the response is equilibrium (or inverted barometer), so that the free surface coincides with Z/g and no *dynamic* currents arise, the effects of mass conservation, ocean self-attraction, and the deformation of the solid earth are held in account. For the purpose of this note, self-consistent may be taken simply to imply $\int_{\text{domain}} Z dx dy = 0$.

together with

$$(x, y) \sim L(x, y), \quad t \sim \omega^{-1}t, \quad f \sim \Omega, \quad \beta \sim \Omega/L,$$

where

$$L \sim (2 \times 10^4/2\pi)\ \text{km}, \quad \omega \sim (2\pi/14)\ \text{d}^{-1},$$
$$\Omega \sim 4\pi\ \text{d}^{-1}, \quad r \ll \omega, \quad H \sim 4000\ \text{m}$$

are appropriate for the large-scale, low-frequency forcing of the fortnightly tide. Scaling η with $\bar{\eta}$ implies we are considering only nonresonant conditions in the sense that the total response deviates only weakly from the equilibrium response. We now determine the relative sizes of η, ψ, and ϕ.

The choice

$$P \sim \frac{\omega L^2 F}{H}$$

results in the nondimensional *continuity equation*,

$$\partial_t \eta = \nabla^2 \phi. \tag{2.2}$$

Since the β term can be taken to dominate the left-hand side of (2.1b), we find that the scaling

$$S \sim P$$

results in the nondimensional *vorticity equation*,

$$\Gamma\psi = \Lambda\phi, \tag{2.3}$$

where

$$\Gamma \equiv \delta\nabla^2\partial_t + \partial_x + \rho\nabla^2, \quad \Lambda \equiv \nabla\cdot(f\nabla),$$
$$\delta \equiv \omega/\Omega, \quad \rho \equiv r/\Omega.$$

Implementing the scalings for ψ and ϕ in (2.1c) yields the nondimensional *divergence equation*,

$$\epsilon\Gamma\phi = -\epsilon\Lambda\psi + \nabla^2\eta - \nabla^2\bar{\eta}, \tag{2.4}$$

where

$$\epsilon \equiv \left(\frac{\Omega^2 L^2}{gH}\right)\left(\frac{\omega}{\Omega}\right) \sim 0.2$$

is a small parameter. (Note that $\epsilon = \gamma\delta$, where γ is a modified Lamb parameter, the square of the ratio of length scale to external Rossby radius, and δ is a familiar small parameter from quasigeostrophic scaling.)

We assume each dependent variable may be expanded asymptotically in powers of ϵ,

$$\xi = \xi^{(0)} + \epsilon\xi^{(1)} + O(\epsilon^2).$$

At zero order, the divergence equation yields $\eta^{(0)} = \bar{\eta}$. Incorporating this result in the zero-order continuity equation then yields the Poisson equation

$$\bar{\eta}_t = \nabla^2\phi^{(0)}, \tag{2.5}$$

with $\nabla\phi^{(0)}\cdot\mathbf{n} = 0$ on the domain boundary. The solution of (2.5) for $\phi^{(0)}$ allows us to solve the zero-order vorticity equation,

$$\Gamma\psi^{(0)} = \Lambda\phi^{(0)}, \tag{2.6}$$

with $\psi^{(0)} = 0$ on the boundary. Together, the zero-order solutions for ψ and ϕ drive the first-order divergence equation,

$$\nabla^2\eta^{(1)} = \Gamma\phi^{(0)} + \Lambda\psi^{(0)}, \tag{2.7}$$

in which $\eta^{(1)}$, the deviation of the free surface from equilibrium, obeys the boundary condition

$$\Delta\eta^{(1)}\cdot\mathbf{n} = 0, \tag{2.8}$$

where

$$\Delta \equiv (f\partial_y + \partial_{xt}^2, f\partial_x - \partial_{yt}^2). \tag{2.9}$$

Equations (2.5), (2.6), and (2.7) thus form a hierarchy of equations for approximating the solution of the dynamic tide, $\eta^{(1)}(x, y)$.

3. Discussion

In dimensional form, the hierarchy of equations for approximating the solution for large-scale, low-frequency nonresonant forced motion is, for the case of periodic forcing proportional to $e^{i\omega t}$,

$$\eta^{(0)} = \bar{\eta} \tag{3.1a}$$

$$\nabla^2\phi^{(0)} = \frac{i\omega}{H}\eta^{(0)} \tag{3.1b}$$

$$(i\omega\nabla^2 + \beta\partial_x + r\nabla^2)\psi^{(0)} = \nabla\cdot(f\nabla\phi^{(0)}) \tag{3.1c}$$

$$g\nabla^2\eta^{(1)} = \nabla\cdot(f\nabla\psi^{(0)}) + i\omega\nabla^2\phi^{(0)} + \beta\phi_x^{(0)} + r\nabla^2\phi^{(0)}. \tag{3.1d}$$

At lowest order, the response is equilibrium (an inverted barometer, in the sense of atmospheric-pressure driving), so that the free surface coincides with the displacement due to the forcing. This nearly equilibrium response then requires, by mass conservation, a divergent velocity field to balance the free-surface displacement. The divergent velocity field drives the vorticity equation via two processes, which we may interpret physically. We rewrite (3.1c), with arbitrary time dependence, as

$$\nabla^2\psi_t^{(0)} + \beta\psi_x^{(0)} + r\nabla^2\psi^{(0)} = \frac{f}{H}\bar{\eta}_t + \beta\phi_y^{(0)}. \tag{3.2}$$

The left-hand side evidently corresponds to "rigid-lid" linear quasigeostrophic dynamics (although we have not invoked the quasigeostrophic approximation). The first forcing term on the right of (3.2) is simply the vortex stretching induced by the equilibrium tide (e.g., Wunsch 1967). The additional term on the right represents β-induced coupling of the divergent velocity field of the time-dependent equilibrium tide. This additional term scales out of the equation if the β-plane approximation, $f = f_0 + \beta y$ for $f_0 \gg \beta L$, is invoked. Note also that, as in unforced quasigeostrophic flow, the *dynamic* divergent velocity field is negligible; the *forced* divergent velocity field, however, is manifestly significant on the planetary scale. O'Connor and Starr (1983) obtain an equation similar to (3.2) from a particular solution for a global pole tide (the oceanic response to the 14-month period Chandler wobble of the earth), but the result was not generalized.

Once the zero-order fields of streamfunction and velocity potential are found from (3.1c), the deviation of the free surface from equilibrium can be obtained from the divergence equation (3.1d). The contribution of $\psi^{(0)}$ to $\eta^{(1)}$ represents a stretched geostrophic balance, the well-known linear balance relation from meteorology (Holten 1979). The contribution of $\phi^{(0)}$ to $\eta^{(1)}$ represents a very weak dynamic divergent velocity signal.

Two questions deserve addressing. Is the result (3.2) simply a consequence of violating the β-plane assumption of quasigeostrophic theory? It is true that the extra forcing term in (3.2) scales out of the equation under the β-plane approximation, but violations of the β-plane approximation are often interpreted as geometric deformations (via slowly varying coefficients) of the significant terms of the quasigeostrophic equations (e.g., Pedlosky 1984). In the case of (3.2), however, forced nonresonant planetary-scale flow can be seen to be influenced by an additional *term* with significant amplitude. For example, if we consider a basin with side lengths, L, centered on the equator ($y = 0$) and choose $\bar{\eta} = A \cos(2\pi y/L)$, which allows $\phi^{(0)}$ to readily satisfy the boundary condition and models the fortnightly equilibrium tidal structure, and let $f(y) = \Omega \sin(\pi y/L)$, then the ratio of the vortex stretching term to the β-coupling term in (3.2) is $\cos^2(\pi y/L)/\cos(2\pi y/L)$, the magnitude of which exceeds unity over much of the basin. Is the assumption of nonresonant, nearly equilibrium flow valid? The fortnightly and monthly tides have been observed to be close to equilibrium (e.g., Wunsch 1967) and time-dependent Sverdrup flow (no Rossby wave excitation) is often suggested to be the dominant low-frequency, wind-driven response of quasigeostrophic models (Willebrand et al. 1980) and of observed flows (Niiler and Koblinsky 1985). Thus, moderate damping at low frequencies appears to effectively extinguish Rossby resonances.

4. Concluding remarks

In summary, if the nonresonant oceanic response to large-scale, low-frequency forcing is expanded about the equilibrium response, (rather than the geostrophic balance as in free-wave expansions) the deviation of the response from equilibrium arises primarily from the vorticity equation. Both vortex stretching and the divergent velocity field of the nearly equilibrium response serve as driving mechanisms for the vorticity equation. Thus, forcing a nonresonant quasigeostrophic model over large meridional spatial scales with, say, fluctuating wind-stress curl alone would be inconsistent, since only vortex stretching would influence the vorticity equation and a significant portion of the forcing would thereby be excluded.

For a *flat-bottomed* ocean, the structure of the β-coupling forcing term in the vorticity equation (3.2) has similar spatial scales as the vortex stretching forcing term. For a *rough-bottomed* ocean, however, the analogous derivation suggests that the forcing function for the vorticity equation may be strongly altered in structure. The significant alteration in forcing function arises from obtaining $\phi^{(0)}$ from the topographic analogue of (3.1b), namely,

$$\nabla \cdot (H \nabla \phi^{(0)}) = i\omega\eta^{(0)}, \qquad (4.1)$$

which may then impose topographic length scales in the β-coupling forcing term of the nearly equilibrium forced vorticity equation. Further work on this topic is in progress.

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